arXiv:math/0306143v1 [math.QA] 9 Jun 2003

The CROCs, non-commutative deformations, and (co)associative bialgebras

Boris Shoikhet

February 1, 2008

Abstract

We compactify the spaces $K(m,n)$ introduced by Maxim Kontsevich. The initial idea was to construct an L_∞ algebra governing the deformations of a (co)associative bialgebra. However, this compactification leads not to a resolution of the PROP of (co)associative bialgebras, but to a new algebraic structure we call here a CROC. It turns out that these constructions are related to the non-commutative deformations of (co)associative bialgebras. We construct an associative dg algebra conjecturally governing the non-commutative deformations of a bialgebra. Then, using the Quillen duality, we construct a dg Lie algebra conjecturally governing the commutative (usual) deformations of a (co)associative bialgebra.

Philosophically, the main point is that for the associative bialgebras the non-commutative deformations is maybe a more fundamental object than the usual commutative ones.

Introduction

The non-commutative deformations of an object, although are not rigorously defined at the moment, exist from a more philosophical point of view. We suppose that the whole formal neighborhood of the moduli space is some formal non-commutative manifold X, while its commutative part is a subspace X_0. It is supposed that in a smooth point, the algebra A of functions on X is a free associative algebra, and the imbedding $X_0 \hookrightarrow X$ gives the corresponding map of algebras $p\colon A \to A_0$ where A_0 is the commutative algebra of functions on X_0. Moreover, we suppose that the map p is the projection $p\colon A \to A/[A,A] = A_0$.

The deformations of an object are described by a dg Lie algebra $\mathfrak{g}^\bullet$ of derivations of an appropriate resolution of the object. On the other hand, this dg Lie algebra is related

to the algebra A_0 in the following way: the 0-th Lie algebra cohomology $H^0(\mathfrak{g}^\bullet) \simeq A_0$ (we do not consider here the questions of completions, etc.). Very informally, we can say that the dg Lie algebra $\mathfrak{g}^\bullet$ is the Quillen dual to the commutative algebra A_0 (in reality, it is not true, it is true only for H^0). Keeping this point of view, it is natural to describe the non-commutative deformations by an object Quillen dual to A. The Quillen duality maps dg commutative algebras to dg Lie algebras and vise versa, and it maps associative dg algebras to itself (this duality is also known as the Koszul duality for operads). This duality associates to an associative algebra A the tensor algebra $\mathrm{T}(A^*[1])$ with the differential $\delta\colon A^* \to (A^*)^{\otimes 2}$ which is dual to the product in A. Then, applying the Quillen construction again, we obtain an associative dg algebra which 0th cohomology is isomorphic to the initial algebra A_0.

On the other hand, the deformation dg Lie algebra $\mathfrak{g}^\bullet$ can be constructed differently from the OPERADic point of view. The well-known example is the deformation theory of associative algebras. The associative algebras itself can be described as algebras over an operad Assoc. Then there is a geometric construction of a free minimal resolution of the operad Assoc. Namely, denote by St_n the n-th Stasheff associahedron (of dimension $n-2$). Then this minimal model $M^\bullet$ is the direct sum $M^\bullet = \bigoplus_{n\geq 2} C_\bullet(\mathrm{St}_n)$ with the chain differential where $C_\bullet$ denotes the chain complex with the Stasheff cell decomposition. By definition, an A_∞ algebra is an algebra over this free operad. In other words, for a vector space V, a map of operads $M^\bullet \to \mathrm{End}(V)$ is the same that an A_∞ algebra structure on V. Consider the tangent space to the space of these maps at some point, denote this tangent space by $\mathrm{Der}(M^\bullet, \mathrm{End}(V))$. Then we can construct from the differential ∂ in $M^\bullet$ an odd vector field on the space $\mathrm{Der}(M^\bullet, \mathrm{End}(V))[-1]$ as we explain in Section 3. In this case of associative algebras, this construction gives exactly the Hochschild complex with the Gerstenhaber bracket. This is the alternative construction of the deformation Lie (in general, L_∞) algebra.

The initial problem from which this paper was grown, is the deformation theory (in the classical, commutative sense) of the (co)associative bialgebras. We tried to construct an L_∞ algebra governing the deformations of an (co)associative bialgebra. We supposed that the underlying complex of this L_∞ algebra is quasi-isomorphic to the Gerstenhaber-Schack complex of the (co)associative bialgebra. We had in mind the operadic construction with the Stasheff associahedrons described above. Then we constructed a compactification of some spaces K_m^n introduced by Maxim Kontsevich. It turned out that the boundary strata of this, a very natural, compatification, are the products of not only the spaces $\mathrm{K}_{m'}^{n'}$, but some spaces with multiindices $\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}$ generalizing the space K_m^n. Then it turned out that the algebraic operations under these spaces form not a PROP as was expected but a new algebraic concept called here a CROC. We construct a CROC $\mathrm{End}(V)$ for a vector space V. Then we have a construction which is a direct generalization of the construction with the minimal model $M^\bullet$ in the Stasheff case, but here this construction gives us an *associative dg algebra*, not a Lie dg algebra. Then we interpret this associative algebra as the associative algebra

governing the *non-commutative* deformations of the initial bialgebra. In the same way we formulate a concept of a non-commutative homotopical bialgebra.

We have analogs of all these constructions in the well-understood case of associative algebras. In this case, we also construct an associative algebra governing the non-commutative deformations. According to the Quillen duality philosophy, we should have "a map" $\mathfrak{g}^\bullet \to (\mathrm{T}(A^*[1]), \delta)$ where the latter is the associative algebra we constructed, and $\mathfrak{g}^\bullet$ is the Hochschild complex with the Gerstenhaber bracket. We construct such a map explicitly.

It happens, however, that in the case of (co)associative bialgebras we *can not* construct such a map. The only what remains is to use again the Quillen duality. We first consider the Quillen dual to the constructed associative algebra, then take its quotient by the commutant. It is a dg commutative algebra, conjecturaly the algebra of functions on the extended commutative moduli space in the formal neighborhood of the point corresponding to the initial bialgebra. Then we construct the Quillen dual dg Lie algebra which conjecturally governs the deformations of the bialgebra.

1 The space $\mathrm{K}(m,n)$ and its compactification

The space $\mathrm{K}(m,n)$ we consider here and its compactification play the same role in the deformation theory of associative bialgebras as the Stasheff polyhedra play in the deformation theory of associative algebras. Here we define this space and construct its compactification $\overline{\mathrm{K}(m,n)}$ which is a (compact) manifold with corners. In the next Sections we define an L_∞-algebra structure on a complex quasi-isomorphic to the Gerstenhaber-Schack complex of an associative bialgebra using this compactification. The spaces $\mathrm{K}(m,n)$ were constructed by Maxim Kontsevich.

1.1 The space $\mathrm{K}(m,n)$.

First define the space $\mathrm{Conf}(m,n)$. By definition, $m,n \geq 1$, $m+n \geq 3$, and

$$\mathrm{Conf}(m,n) = \{p_1,\dots,p_m \in \mathbb{R}^{(1)}, p_i < p_j \;\; for \;\; i<j; \\ q_1,\dots,q_n \in \mathbb{R}^{(2)}, q_i < q_j \;\; for \;\; i<j\} \quad (1)$$

Here we denote by $\mathbb{R}^{(1)}$ and by $\mathbb{R}^{(2)}$ two different copies of a real line $\mathbb{R}$.

Next, define a 3-dimensional group G^3 acting on $\mathrm{Conf}(m,n)$. This group is a semidirect product $G^3 = \mathbb{R}^2 \ltimes \mathbb{R}_+$ (here $\mathbb{R}_+ = \{x \in \mathbb{R}, x > 0\}$) with the following group law:

$$(a,b,\lambda)\circ(a',b',\lambda') = (\lambda' a + a', (\lambda')^{-1} b + b', \lambda\lambda') \quad (2)$$

where $a,b,a',b' \in \mathbb{R}, \lambda,\lambda' \in \mathbb{R}_+$. This group acts on the space $\mathrm{Conf}(m,n)$ as

$$(a,b,\lambda)\cdot(p_1,\dots,p_m;q_1,\dots,q_n) = (\lambda p_1 + a,\dots,\lambda p_m + a; \lambda^{-1}q_1 + b,\dots,\lambda^{-1}q_n + b) \quad (3)$$

In other words, we have two independent shifts on $\mathbb{R}^{(1)}$ and $\mathbb{R}^{(2)}$ (by a and b), and $\mathbb{R}_+$ dilatates $\mathbb{R}^{(1)}$ by λ and dilatates $\mathbb{R}^{(2)}$ by λ^{-1}.

In our conditions $m, n \geq 1, m+n \geq 3$, the group G^3 acts on $\mathrm{Conf}(m,n)$ freely. Denote by $\mathrm{K}(m,n)$ the quotient-space. It is a smooth manifold of dimension $m+n-3$.

1.1.1 Example

Let $m = n = 2$. Then the space $\mathrm{K}(2,2)$ is 1-dimensional. It is easy to see that $(p_2 - p_1) \cdot (q_2 - q_1)$ is preserved by the action of G^3, and it is the only invariant of the G^3-action on $\mathrm{K}(2,2)$. Therefore, $\mathrm{K}(2,2) \simeq \mathbb{R}_+$. There are two "limit" configurations: $(p_2 - p_1) \cdot (q_2 - q_1) \to 0$ and $(p_2 - p_1) \cdot (q_2 - q_1) \to \infty$. Therefore, the compactification $\overline{\mathrm{K}(2,2)} \simeq [0,1]$.

The main trouble in the problem of constructing the (right) compactification $\overline{\mathrm{K}(m,n)}$ in the general case is that the space $\mathrm{K}(m,n)$ is not compact, and the points can move away from each other on infinite distances. Moreover, all these infinities are not the same, in particular, ∞ and ∞^2 are different infinities when they occur in the same configuration. See the following example.

1.1.2 Example

Consider the space $\mathrm{K}(1,n)$, $n \geq 2$. Consider the following "limit" configuration in $\mathrm{K}(n,1)$: first two points $q_1, q_2 \in \mathbb{R}^{(2)}$ are in a finite distance from each other; the point q_3 is in the distance ∞ from q_1, q_2; the point q_4 is in the distance ∞^2 from q_3; q_5 is in the distance ∞^3 from q_4, and so on. We will see in the next Subsection that the space of all such configurations in $\mathrm{K}(1,n)$ has dimension 0.

Now we are going to construct the compactification $\overline{\mathrm{K}(m,n)}$ in the general case.

1.2 The compactification $\overline{\mathrm{K}(m,n)}$.

1.2.1 The space $\mathrm{K}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}$.

Define first the space $\mathrm{K}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}$ of dimension $\sum_{i=1}^{\ell_1} m_i + \sum_{i=1}^{\ell_2} n_i - \ell_1 - \ell_2 - 1$ (here $m_i, n_i \geq 1$ and $\sum_{i=1}^{\ell_1} m_i + \sum_{i=1}^{\ell_2} n_i \geq \ell_1 + \ell_2 + 1$). The space $\mathrm{K}(m,n)$ a particular case of these spaces, when $\ell_1 = \ell_2 = 1$, $\mathrm{K}(m,n) = \mathrm{K}^m_n$. Fist define the space $\mathrm{Conf}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}$. By definition,

$$\begin{aligned}
\mathrm{Conf}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}} = \\
&\{p^1_1,\dots,p^1_{m_1} \in \mathbb{R}^{(1,1)}, p^2_1,\dots,p^2_{m_2} \in \mathbb{R}^{(1,2)},\dots,p^{\ell_1}_1,\dots,p^{\ell_1}_{m_{\ell_1}} \in \mathbb{R}^{(1,\ell_1)};\\
&\quad q^1_1,\dots,q^1_{n_1} \in \mathbb{R}^{(2,1)}, q^2_1,\dots,q^2_{n_2} \in \mathbb{R}^{(2,2)}\dots,q^{\ell_2}_1,\dots,q^{\ell_2}_{n_{\ell_2}} \in \mathbb{R}^{(2,\ell_2)} |\\
&\qquad p^j_{i_1} < p^j_{i_2} \quad for \quad i_1 < i_2; q^j_{i_1} < q^j_{i_2} \quad for \quad i_1 < i_2\} \qquad (4)
\end{aligned}$$

Here $\mathbb{R}^{(i,j)}$ are copies of the real line $\mathbb{R}$. Now we have an $\ell_1+\ell_2+1$-dimensional group $G^{\ell_1,\ell_2,1}$ acting on $\mathrm{Conf}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}$. It contains $\ell_1+\ell_2$ independent shifts

$$p_i^j\mapsto p_i^j+a_j, i=1,\dots,m_j, a_j\in\mathbb{R}; q_i^j\mapsto q_i^j+b_j, i=1,\dots,n_j, b_j\in\mathbb{R}$$

and *one* dilatation

$$p_i^j\mapsto \lambda\cdot p_i^j \ \ for \ \ all \ \ i,j; q_i^j\mapsto \lambda^{-1}\cdot q_i^j \ \ for \ \ all \ \ i,j.$$

This group is isomorphic to $\mathbb{R}^{\ell_1+\ell_2}\ltimes\mathbb{R}_+$. We say that the lines $\mathbb{R}^{(1,1)},\mathbb{R}^{(1,2)},\dots,\mathbb{R}^{(1,\ell_1)}$ (corresponding to the factor λ) are the lines of the first type, and the lines $\mathbb{R}^{(2,1)},\mathbb{R}^{(2,2)},\dots,\mathbb{R}^{(2,\ell_2)}$ (corresponding to the factor λ^{-1}) are the lines of the second type.

Denote

$$\mathrm{K}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}=\mathrm{Conf}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}/G^{\ell_1,\ell_2,1} \tag{5}$$

The strata of the compactification $\overline{\mathrm{K}(m,n)}$ constructed below are direct products of spaces $\mathrm{K}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}$ for different ℓ_1,ℓ_2,m_i,n_j.

1.2.2 `The Construction.`

We define here a stratified manifold $\overline{\mathrm{K}(m,n)}$.

Suppose we have a "limit" configuration $\sigma\in\mathrm{K}(m,n)$; "limit" here means that some distances $|p_i-p_j|$ or $|q_i-q_j|$ are infinitely small or infinitely large. In the sequel we say "equal to 0" and "equal to ∞" in these cases. The non-limit configurations form the maximal open stratum isomorphic to $\mathrm{K}(m,n)$.

Each limit configuration belongs to a unique stratum of the form

$$\mathrm{K}(\sigma)\simeq\mathrm{K}^{m_1,\dots,m_{\ell_1}}_{n_1,\dots,n_{\ell_2}}\times\mathrm{K}^{m'_1,\dots,m'_{\ell'_1}}_{n'_1,\dots,n'_{\ell'_2}}\times\dots \tag{6}$$

which we are going to describe.

Consider the set of all (finite, infinitely small, or infinitely large) parameters λ such that the image σ_λ of σ after the application of the element $(0,0,\lambda)\in G^3$obeys the following property (*):

> `In` σ_λ **`either`** `there exist at least two points` p_i `and` p_{i+1} `in` $\mathbb{R}^{(1)}$ `such that the distance` $|p_{i+1}-p_i|$ `is` **`finite`** `(nor infinitely small neither infinitely large)` **`or`** `at least two points` q_j,q_{j+1} `in` $\mathbb{R}^{(2)}$ `with the same property for` $|q_{j+1}-q_j|$.

We say that two parameters λ_1,λ_2 obeying the property (*) are equivalent, if the ratio $\frac{\lambda_1}{\lambda_2}$ is finite and not infinitesimally small.

Denote by $S(\sigma)$ the set of the equivalence classes of the parameters λ obeying the property (*) for the configuration σ. The set $S(\sigma)$clearly is not empty for any σ, and the condition $\sharp S(\sigma)=1$ is equivalent that σ is a non-limit configuration. For a limit σ, $\sharp S(\sigma)>1$. It is clear that for any σ the set $S(\sigma)$ is finite.

Example. Consider the configuration σ from Example 1.1.2. We have the space $\mathrm{K}(1,n)$, and $|q_2 - q_1|$ is finite, $|q_3 - q_2| \sim \infty$, $|q_4 - q_3| \sim \infty^2$, $\dots, |q_n - q_{n-1}| \sim \infty^{n-2}$. We have: $\sharp S(\sigma) = n-1$. Indeed, roughly speaking, $\lambda_1 = 1, \lambda_2 = \infty, \dots, \lambda_{n-1} = \infty^{n-2}$obey the property (*), it is clear that each λ obeying the property (*) for σ is equivalent to some λ_i from the list above.

Remark. We should specify what is meant by a limit configuration. Each point on each line, when it moves, becomes a real-valued function on a real parameter t. Thus, we have functions $p_1(t), \dots, p_m(t); q_1(t), \dots, q_n(t)$. *We suppose that all these functions are Lourent power series in the parameter t.* Then, a limit configuration is this configuration when $t \to \infty$. It is important that we do not consider some more other functions in t except polynomials in t, t^{-1}. In a sense, thus we obtain the minimal compactification.

Consider the configuration $(0,0,\lambda)\cdot\sigma = \sigma_\lambda$. We identify in σ_λ any two points which are infinitely close to each other. Then the equivalence classes (under this identification) of the points on $\mathbb{R}^{(1)}$ in σ_λ can be uniquely divided to $\ell_1(\lambda)$ groups

$$\{p_1^1, \dots, p^1_{m_1(\lambda)}\}, \{p_1^2, \dots, p^2_{m_2(\lambda)}\}, \dots, \{p_1^{\ell_1(\lambda)}, \dots, p^{\ell_1(\lambda)}_{m_{\ell_1(\lambda)}(\lambda)}\}$$

of points standing in turn such that inside each group all the distances between the points are finite (and nonzero, because we have collapsed the points infinitely close to each other), and the distances between different groups are ∞. Analogously, we divide the points on $\mathbb{R}^{(2)}$ in σ_λ to the $\ell_2(\lambda)$ groups

$$\{q_1^1, \dots, p^1_{n_1(\lambda)}\}, \{q_1^2, \dots, q^2_{n_2(\lambda)}\}, \dots, \{q_1^{\ell_2(\lambda)}, \dots, q^{\ell_2(\lambda)}_{n_{\ell_2(\lambda)}(\lambda)}\}$$

by the same way.

We associate with the element $\lambda \in S(\sigma)$ the space

$$\mathrm{K}(\sigma_\lambda) = \mathrm{K}^{m_1(\lambda),\dots,m_{\ell_1(\lambda)}(\lambda)}_{n_1(\lambda),\dots,n_{\ell_2(\lambda)}(\lambda)} \tag{7}$$

Definition. We say that two limit configurations σ_1, σ_2 are equivalent iff:

(i) the sets $S(\sigma_1)$ and $S(\sigma_2)$ are **coincide** (it means that $\sharp S(\sigma_1) = \sharp S(\sigma_2)$, and we can choose representatives $\lambda_1, \dots, \lambda_{\sharp S(\sigma_1)}$ for $S(\sigma_1)$, and representatives $\lambda'_1, \dots, \lambda'_{\sharp S(\sigma_2)}$ for $S(\sigma_2)$ such that there exists λ_{tot} (finite, infinitely small, or infinitely large) such that $\lambda'_i = \lambda_{tot}\cdot\lambda_i$ for each i);

(ii) the sets $\{\{\ell_1(\lambda)\}, \{\ell_2(\lambda)\}, \{m_1(\lambda), \dots, m_{\ell_1(\lambda)}(\lambda)\}, \{n_1(\lambda), \dots, n_{\ell_2(\lambda)}(\lambda)\} | \lambda \in S(\sigma_i)\}$ coincide as ordered sets under the identification with λ_{tot} described in (i).

In other words, two limit configurations are equivalent if their "topological types" coincide after the application of some $(0,0,\lambda_{tot}) \in G^3$ for finite, infinitely small, or infinitely large λ_{tot}.

Theorem. *(i) For a configuration σ (limit or non-limit) with m points on $\mathbb{R}^{(1)}$ and n points on $\mathbb{R}^{(2)}$, the space of all configurations equivalent to σ is homeomorphic to*

$$\mathrm{K}(\sigma) = \prod_{[\lambda]\in S(\sigma)} \mathrm{K}(\sigma_\lambda) \tag{8}$$

(see (7) for the definition of $\mathrm{K}(\sigma_\lambda)$);

(ii)

$$\overline{\mathrm{K}(m,n)} := \bigsqcup_{\substack{\text{all equiv.}\\ \text{classes of}\\ \text{config. } \sigma}} \mathrm{K}(\sigma) \tag{9}$$

defines a compactification of the space $\mathrm{K}(m,n)$; it is a manifold with corners.

Proof. it is clear. The only what we want to notice is that the distances between the groups $\{p_1^1, \dots, p^1_{m_1(\lambda)}\}, \dots$ are ∞,therefore, their positions on the lines are defined up to shifts, for a shift for each group. It motivates our definition of the space $\mathrm{K}(\sigma_\lambda)$. $\square$

Corollary. *For each σ, $\dim \mathrm{K}(\sigma) \le m+n-3$.* $\square$

Although the Corollary follows from Theorem above, it is instructive to give here a straightforward proof of it. We want to prove that $\dim \mathrm{K}(\sigma) \le \dim \mathrm{K}(m,n) = m+n-3$, and the equality holds only for a non-limit configuration σ.

The set $S(\sigma)$ is naturally ordered by the numbers $\lambda \in \mathbb{R}_+$ representing the equivalence class. Consider the minimal element $[\lambda] \in S(\sigma)$. Then there are no 0 distances between points on $\mathbb{R}^{(2)}$ in σ_λ. Then the points in $\mathbb{R}^{(2)}$ in the configuration σ_λ are divided to the groups $\{q_1^1, \dots, q^1_{n_1(\lambda)}\}, \dots, \{q_1^{\ell_2(\lambda)}, \dots, q^{\ell_2(\lambda)}_{n_{\ell_2(\lambda)}(\lambda)}\}$, and also the points on $\mathbb{R}^{(1)}$ are divided to the groups (but there can occur 0 distances). We will consider only that part of $\dim \mathrm{K}(\sigma)$ which is contributed by the points on $\mathbb{R}^{(2)}$, the contribution of points on $\mathbb{R}^{(1)}$ is analogous. We denote this dimension by $\dim_2$. Thus, for the minimal $\lambda \in S(\sigma)$, $\dim_2 \mathrm{K}(\sigma_\lambda) = n_1(\lambda) + \dots + n_{\ell_2(\lambda)}(\lambda) - \ell_2(\lambda)$. Also we have $\dim \mathrm{K}(\sigma_\lambda) = \dim_1 \mathrm{K}(\sigma_\lambda) + \dim_2 \mathrm{K}(\sigma_\lambda) - 1$ (the last -1 because of the action of $(0,0,\mathbb{R}_+) \in G^{\ell_1,\ell_2,1}$). Now consider the next (in the sense of the canonical ordering) element $\lambda' \in S(\sigma)$. It is clear that $\frac{\lambda'}{\lambda} = \infty$. Each group of points $\{q_1^i, \dots, q^i_{n_i(\lambda)}\}$ will be collapsed to a point $\overline{q_{n_i}}$ in $\sigma_{\lambda'}$, and only these groups will be collapsed because of our choice of λ'. Then, in $\sigma_{\lambda'}$ the points $\overline{q_1}, \dots, \overline{q_{\ell_2(\lambda)}}$ are divided to $\ell_2(\lambda')$ groups, and

$$\dim_2 \mathrm{K}(\sigma_{\lambda'}) = \ell_2(\lambda) - \ell_2(\lambda')$$

We see that

$$\dim_2 \mathrm{K}(\sigma_\lambda) + \dim_2 \mathrm{K}(\sigma_{\lambda'}) = n_1(\lambda) + \dots + n_{\ell_2(\lambda)} - \ell_2(\lambda') = n - \ell_2(\lambda')$$

We can show analogously, that after d steps,

$$\dim_2 \mathrm{K}(\sigma_{\lambda_1}) \times \cdots \times \mathrm{K}(\sigma_{\lambda_d}) = n - \ell_2(\lambda_d)$$

(here $\lambda_1 < \cdots < \lambda_d$ are the first lowest d elements in $S(\sigma)$). For the maximal $\lambda_{max} \in S(\sigma)$, $\ell_2(\lambda_{max}) = 1$ (there is the only 1 group of points). Therefore, $\dim_2 \mathrm{K}(\sigma) = n - 1$. It proves the following

Proposition. $\dim \mathrm{K}(\sigma) = m + n - \sharp S(\sigma) - 2$ $\square$

Remark. In the case $m = 1$ the space $\overline{\mathrm{K}(1,n)}$ is *NOT* the Stasheff polyhedron. We can see it immediately. In particular, in our compactification it is important the "relative velocity" with which points move close to each other. In fact, in the Stasheff compactification it is irrelevant. This fact could hint us that the algebraic structures behind our compactification is not anymore some usual structures like OPERADs and PROPs.

1.2.3 Examples

1.2.3.1 **Example** We already know that the space $\mathrm{K}(2,2)$ has 2 different limit configurations: they are σ_1, when $|p_2 - p_1| \cdot |q_2 - q_1| \sim 0$, and σ_2, when $|p_2 - p_1| \cdot |q_2 - q_1| \sim \infty$ (see Example 1.1.1).We have:

$$\mathrm{K}(\sigma_1) = \mathrm{K}_2^1 \times \mathrm{K}_1^2 \tag{10}$$

$$\mathrm{K}(\sigma_2) = \mathrm{K}_2^{1,1} \times \mathrm{K}_{1,1}^2 \tag{11}$$

In both cases $\dim \mathrm{K}(\sigma_i) = 0$.

1.2.3.2 **Example** Consider the limit configuration σ in the space $\overline{\mathrm{K}(1,n)}$ described in the Example 1.1.2. Recall that for this configuration σ we have: $|q_2 - q_1|$ is finite, $|q_3 - q_2| \sim \infty$, $|q_4 - q_3| \sim \infty^2$,..., $|q_n - q_{n-1}| \sim \infty^{n-2}$. We have: $\sharp S(\sigma) = n - 1$, and therefore $\dim \mathrm{K}(\sigma) = 0$ by Proposition 1.2.2. We have:

$$\mathrm{K}(\sigma) = \mathrm{K}^1_{2,1,\dots,1(n-2\ of\ 1's)} \times \mathrm{K}^1_{2,1,\dots,1(n-3\ of\ 1's)} \times \cdots \times \mathrm{K}^1_2.$$

1.3 Strata of codimension 1

Here we describe all strata of codimension 1 in $\overline{\mathrm{K}(m,n)}$. These strata are very important in the next Sections where we construct an $L-\infty$-algebra structure on the deformation complex of an associative bialgebra.

A typical stratum of codimension 1 is drawn in Figure 1.

Consider numbers ℓ_1, ℓ_2, $\{m_1, \dots, m_{\ell_1}\}$, $\{n_1, \dots, n_{\ell_2}\}$ satisfying $\sum_{i=1}^{\ell_1} m_i = m$, $\sum_{i=1}^{\ell_2} n_i = n$. Consider the limit configuration $\sigma \in \overline{\mathrm{K}(m,n)}$ in which the points on $\mathbb{R}^{(1)}$ are divided to ℓ_1 groups, m_i points in the i-th group, and the points inside each

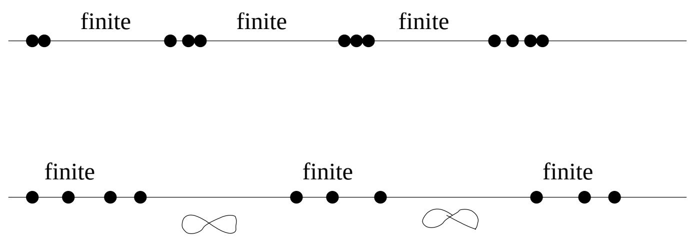

Figure 1: A typical stratum of codimension 1

group are infinitely close to each other (in order ε); the points on $\mathbb{R}^{(2)}$ are divided to ℓ_2 groups, n_j points in the j-th group, and points inside each group are in finite distance from each other, and the points of different groups are infinitely far from each other (in order $\frac{1}{\varepsilon}$). It is clear that $\sharp S(\sigma) = 2$ (representatives in $S(\sigma)$ are $\lambda = 1$ and $\lambda = \frac{1}{\varepsilon}$), and it is the most general configuration with $\sharp S(\sigma) = 2$. Then it follows from Proposition 1.2.2 that these configurations σ exhaust all strata of codimension 1.

1.4 The space $\overline{\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$

Here we construct the compactification $\overline{\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$ of the space $\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}$ which we will need in the sequel.

The construction is analogous to the construction of $\overline{\mathrm{K}(m,n)}$ above. We associate with a configuration in $\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}$ a limit configuration in $\mathrm{K}(\sum_{i=1}^{\ell_1} m_i, \sum_{j=1}^{\ell_2} n_j)$. Namely, we divide the $\sum_{i=1}^{\ell_1} m_i$ points into ℓ_1 groups with m_i points in the i-th group, and do the same with the second line. We suppose that the distances between groups are ∞^N where $N \gg 0$. In other words, we suppose that the distances between the groups are infinitely large comparably with all other infinities in the (sub)limit configurations we consider. Then the previous construction can be easily generalized to this case.

2 The concept of CROC

Here we introduce the concept of *CROC*. This notion formalizes the admissible operations on an algebraic structure, like OPERADs and PROPs. We tried to formalize operations which we have among the spaces $\overline{\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$ which is an example of a topological CROC. It turns out that we can describe the associative bialgebras as algebras over some CROC, and this CROC of associative bialgebras has a very natural simplicial free minimal model, analogous to the Stasheff construction in the case of associative algebras. This simplicial free resolution is formed from the chain complexes of the spaces $\overline{\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$.

2.1 The Definition

Definition. A preCROC of vector spaces is a collection of vector spaces $F^{n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}}$, $m_i, n_j \geq 1$ with a left action of the product of symmetric groups $\Sigma_{m_1}\times\dots\times\Sigma_{m_{\ell_1}}$ and a right action of $\Sigma_{n_1}\times\dots\times\Sigma_{n_{\ell_2}}$ and with a composition law. In the simplest case, this composition law is a map

$$\phi^{\ell_1|n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}|\ell_2}\colon F^{\ell_2}_{m_1,\dots,m_{\ell_1}}\otimes F^{n_1,\dots,n_{\ell_2}}_{\ell_1}\to F^{n_1+n_2+\dots+n_{\ell_2}}_{m_1+m_2+\dots+m_{\ell_1}} \tag{12}$$

In general, we have the composition law

$$\phi^{\ell_1^1,\ell_1^2,\dots,\ell_1^a|n_1^1,\dots,n^1_{\ell_2^1};\dots;n_1^b,\dots,n^b_{\ell_2^b}}_{m_1^1,\dots,m^1_{\ell_1^1};\dots;m_1^a,\dots,m^a_{\ell_1^a}|\ell_2^1,\dots,\ell_2^b}\colon F^{\ell_2^1,\dots,\ell_2^b}_{m_1^1,\dots,m^1_{\ell_1^1},\dots,m_1^a,\dots,m^a_{\ell_1^a}}\otimes F^{n_1^1,\dots,n^1_{\ell_2^1},\dots,n_1^b,\dots,n^b_{\ell_2^b}}_{\ell_1^1,\dots,\ell_1^a}\to$$
$$F^{\sum_{j=1}^{\ell_2^1}n_j^1,\dots,\sum_{j=1}^{\ell_2^b}n_j^b}_{\sum_{i=1}^{\ell_1^1}m_i^1,\dots,\sum_{i=1}^{\ell_1^a}m_i^a} \tag{13}$$

There are three axioms on this data:

(i) F^1_1 is the 1-dimensional trivial representation of $\Sigma_1\times\Sigma_1$;

(ii) the compositions are compatible with the action of symmetric groups;

(iii) the natural associativity of the compositions.

Definition. A CROC is a preCROC with the following extra conditions:

(i) there are the *restriction maps* $r_i\colon F^{n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}}\to F^{n_1,\dots,n_{\ell_2}}_{m_1,\dots,\hat{m}_i,\dots,m_{\ell_1}}$ and $r^j\colon F^{n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}}\to F^{n_1,\dots,\hat{n}_j,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}}$,

(ii) the restrictions satisfy the natural commutativity of the compositions.

We will consider algebras over (pre)CROCs. For this we define the preCROC End(V) and define an algebra over a preCROC F structure on a vector space V as a map of preCROCs $F\to\mathrm{End}(V)$. Note that the definition of the preCROC End(V) is not very straightforward, we do it in the next Subsection.

2.2 The preCROC End(V)

Let V be a vector space. Here we define the CROC End(V). By definition,

$$\mathrm{End}(V)^{n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}}=\bigotimes_{i=1\dots\ell_1,j=1\dots\ell_2}\mathrm{Hom}(V^{\otimes m_i},V^{\otimes n_j}) \tag{14}$$

We should define now the composition and the maps ϑ_i, ϑ^i. First define the composition.

We define first the simplest composition

$$\phi^{\ell_1|n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}|\ell_2}\colon \mathrm{End}(V)^{\ell_2}_{m_1,\dots,m_{\ell_1}} \otimes \mathrm{End}(V)^{n_1,\dots,n_{\ell_2}}_{\ell_1} \to \mathrm{End}(V)^{n_1+n_2+\dots+n_{\ell_2}}_{m_1+m_2+\dots+m_{\ell_1}}$$

Suppose we have

$$\Psi_1 \in \mathrm{Hom}(V^{\otimes \ell_1}, V^{\otimes n_1}), \Psi_2 \in \mathrm{Hom}(V^{\otimes \ell_1}, V^{\otimes n_2}), \dots, \Psi_{\ell_2} \in \mathrm{Hom}(V^{\otimes \ell_1}, V^{\otimes n_{\ell_2}})$$

and

$$\Theta_1 \in \mathrm{Hom}(V^{\otimes m_1}, V^{\otimes \ell_2}), \Theta_2 \in \mathrm{Hom}(V^{\otimes m_2}, V^{\otimes \ell_2}), \dots, \Theta_{\ell_1} \in \mathrm{Hom}(V^{\otimes m_{\ell_1}}, V^{\otimes \ell_2}),$$

we are going to define their composition which belongs to $\mathrm{End}(V)^{n_1+\dots+n_{\ell_2}}_{m_1+\dots+m_{\ell_1}}$. Denote $m = m_1 + \dots + m_{\ell_1}$, $n = n_1 + \dots + m_{\ell_2}$. The construction is as follows:

First define

$$F(v_1 \otimes \dots \otimes v_m)\colon = \Theta_1(v_1 \otimes \dots \otimes v_{m_1}) \bigotimes \Theta_2(v_{m_1+1} \otimes \dots \otimes v_{m_1+m_2}) \bigotimes \dots \\ \bigotimes \Psi_{\ell_2}(v_{m_1+\dots+m_{\ell_1 -1}+1} \otimes \dots \otimes v_{m_1+\dots+m_{\ell_1}}) \in V^{\otimes \ell_1 \ell_2} \quad (15)$$

Now we apply $\{\Theta_i\}$ to this element in $V^{\otimes \ell_1\ell_2}$: we define an element $G\colon V^{\otimes \ell_1\ell_2} \to V^{\otimes m}$ as follows:

$$G(v_1 \otimes v_2 \otimes \dots \otimes v_{\ell_1\ell_2}) := \Theta_1(v_1 \otimes v_{\ell_2+1} \otimes \dots \otimes v_{\ell_2(\ell_1-1)+1}) \bigotimes \\ \Theta_2(v_2 \otimes v_{\ell_2+2} \otimes \dots \otimes v_{\ell_2(\ell_1-1)+2}) \bigotimes \dots \bigotimes \Theta_{\ell_1}(v_{\ell_2} \otimes v_{2\ell_2} \otimes \dots \otimes v_{\ell_2\ell_1}) \in V^{\otimes n}. \quad (16)$$

Define now

$$Q(v_1 \otimes \dots \otimes v_m) := G \circ F(v_1 \otimes \dots \otimes v_m) \quad (17)$$

By definition, the element Q is the composition $\phi^{\ell_1|n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}|\ell_2}(\Theta_1 \otimes \dots \otimes \Theta_{\ell_1} \otimes \Psi_1 \otimes \dots \otimes \Psi_{\ell_2})$

This construction can be easily generalized to the higher compositions $\phi^{\ell_1^1,\ell_1^2,\dots,\ell_1^a|n_1^1,\dots,n^1_{\ell_2^1};\dots;n_1^b,\dots,n^b_{\ell_2^b}}_{m_1^1,\dots,m^1_{\ell_1^1};\dots;m_1^a,\dots,m^a_{\ell_1^a}|\ell_2^1,\dots,\ell_2^b}$ such that the associativity holds. The formulas are very huge but the pictures behind them are very simple.

We can easily prove that the structure defined in this way is a preCROC.

Definition. (i) `Let` F `be a preCROC. Then an` F`-algebra structure on a vector space` V `is a map of preCROCs` $F \to \mathrm{End}(V)$`,`

`(ii) Let` F `be a CROC. Then an` F`-algebra on a vector space` V `is a map of preCROCs` $\Upsilon\colon F \to \mathrm{End}(V)$ `such that` $\Upsilon(F^1_1) = \mathrm{Id} \in \mathrm{Hom}(V, V)$ `and which is`

compatible with the restrictions on F and with the tensor product on $\mathrm{End}(V)$:

$$\Upsilon(F^{n_1,\dots,n_{\ell_1}}_{m_1,\dots,m_{\ell_1}}) = \Upsilon(r_i(F^{n_1,\dots,n_{\ell_1}}_{m_1,\dots,m_{\ell_1}})) \otimes \Upsilon(\bar{r_i}(F^{n_1,\dots,n_{\ell_1}}_{m_1,\dots,m_{\ell_1}})) \tag{18}$$

where $\bar{r_i}\colon F^{n_1,\dots,n_{\ell_1}}_{m_1,\dots,m_{\ell_1}} \to F^{n_1,\dots,n_{\ell_1}}_{m_i}$ is the "complementary" restriction which is defined as the composition of the corresponding restrictions. The same should be true for r^j.

2.3 The CROC of associative bialgebras

In this Subsection we define the CROC Assoc of associative bialgebras and show that a map of CROCs $\mathrm{Assoc} \to \mathrm{End}(V)$ is the same that an associative bialgebra structure on V (here by an associative bialgebra we mean a Hopf algebra without the unit, the counit, and the antipode, that is, it has an associative product, a coassociative coproduct, which are compatible).

By definition, $\mathrm{Assoc}^{n_1,\dots,n_{\ell_2}}_{m_1,\dots,m_{\ell_1}} = \prod_{i=1\dots\ell_1, j=1\dots\ell_2} \Sigma_{m_i} \times \Sigma_{n_j}$ where Σ_i is the symmetric group. Now we are going to define the compositions.

For this, for any associative bialgebra V we associate with an element in $\Sigma_{m_i} \times \Sigma_{n_j}$ the following element in $\mathrm{Hom}(V^{\otimes m_i}, V^{\otimes n_j})$: $\Psi_{i,j}\colon v_1 \otimes \dots \otimes v_{m_i} \mapsto \sigma^{(2)}_{i,j} \circ \Delta^{n_j-1}(v_{\sigma^{(1)}_{i,j}1} \star \dots \star v_{\sigma^{(1)}_{i,j}m_i})$ where we denote by $\sigma^{(1)}_{i,j}, \sigma^{(2)}_{i,j}$ the corresponding permutations from the symmetric groups. Thus, we attached to each element in $\Sigma_{m_i} \times \Sigma_{n_j}$ an element in $\mathrm{Hom}(V^{\otimes m_i}, V^{\otimes n_j})$ for any bialgebra V. We claim that there exists a unique CROC structure on Assoc such that for any bialgebra V the constructed map is a map of CROCs $\mathrm{Assoc} \to \mathrm{End}(V)$. Indeed, the composition of the corresponding Hom's is again a homomorphism of this form because of the associativity, the coassociativity, and the compatibility of the product with the coproduct. We can write down the corresponding permutation by an explicit formula. We do not do that because this formula will not tell us anything new.

Now we are going to prove the following result:

Theorem. *Any map of CROCs $\Upsilon\colon \mathrm{Assoc} \to \mathrm{End}(V)$ is equivalent to the map described above for some bialgebra structure on V.*

Proof. We already shown that any bialgebra structure on V, by the definition of the CROC Assoc, gives a map of CROCs $\mathrm{Assoc} \to \mathrm{End}(V)$. To prove the reverse statement, first denote by $\Psi = \Upsilon(\mathrm{Assoc}^1_2) \in \mathrm{Hom}(V^{\otimes 2}, V)$ and $\Delta = \Upsilon(\mathrm{Assoc}^2_1) \in \mathrm{Hom}(V, V^{\otimes 2})$. We want to prove the associativity for Ψ, the coassociativity for Δ, and their compatibility. To prove say the associativity, consider the maps (CROC's compositions) $i_1\colon \mathrm{Assoc}^1_{1,2} \times \mathrm{Assoc}^1_2 \to \mathrm{Assoc}^1_3$ and $i_2\colon \mathrm{Assoc}^1_{2,1} \times \mathrm{Assoc}^1_2 \to \mathrm{Assoc}^1_3$ It is clear that they coincide when applied to the identity elements of the symmetric groups. Then, as we have a map of CROCs, the CROC compositions of their images also should coincide. We prove the coassociativity in the same way. To prove the compatibility, note that the two maps

$t_1\colon \mathrm{Assoc}_2^1 \times \mathrm{Assoc}_1^2 \to \mathrm{Assoc}_2^2$ and $t_2\colon \mathrm{Assoc}_{1,1}^2 \times \mathrm{Assoc}_2^{1,1} \to \mathrm{Assoc}_2^2$ coincide on the identity elements of the symmetric groups. Then, using the factorization (the property (ii) of the Definition above) we get the claim. $\square$

2.4 A free resolution of the CROC Assoc

Consider the direct sum of all chain complexes $\aleph = \bigoplus_{m_1,\dots,m_{\ell_1};n_1,\dots n_{\ell_2}} C_\bullet \overline{K_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$. As $\{\overline{K_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}\}$ form a topological CROC, $\aleph$ is a CROC of graded vector spaces, namely, $\aleph_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}} = C_\bullet \overline{K_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$. Moreover, we have a differential (the chain differential) on this dg CROC. We can prove the following theorem:

Theorem. *(i) The CROC $\aleph$, as a dg CROC, is free,*

(ii) The cohomology of the CROC $\aleph$ is isomorphic to the CROC Assoc*.*

Proof. (i) is clear, (ii) follows from the fact that all spaces $\overline{K_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}}}$ are contractible. $\square$

2.5 The algebra $\mho$

Here we define our main object–an associative algebra $\mho$. As a vector space,

$$\mho = \bigoplus_{m_1,\dots,m_{\ell_1}\ge 1, n_1,\dots,n_{\ell_2}\ge 1} \left(\bigotimes_{1\le i\le \ell_1, 1\le j\le \ell_2} \mathrm{Hom}(V^{\otimes m_i}, V^{\otimes n_j}) \right) [-\sum m_i - \sum n_j + \ell_1 + \ell_2] \tag{19}$$

Note that the grading is compatible with the grading in the Gerstenhaber-Schack complex. Now we define an associative product on $\mho$. Let $\Psi_1 \in \left(\bigotimes_{1\le i\le \ell_1, 1\le j\le \ell_2} \mathrm{Hom}(V^{\otimes m_i}, V^{\otimes n_j})\right)[-\sum m_i - \sum n_j + \ell_1 + \ell_2]$ and $\Psi_2 \in \left(\bigotimes_{1\le i\le \ell_1', 1\le j\le \ell_2'} \mathrm{Hom}(V^{\otimes m_i'}, V^{\otimes n_j'})\right)[-\sum m_i' - \sum n_j' + \ell_1' + \ell_2']$. Define their product as the composition in the preCROC $\mathrm{End}(V)$ (it is 0, if the corresponding composition $\phi_{\dots}^{\dots}$ in the preCROC $\mathrm{End}(V)$ is 0) *up to a sign.* This sign is defined geometrically from the boundary operator in $\aleph$ as follows.

For Ψ_1 and Ψ_2 as above, if their product is nonzero, there exist *a unique* stratum of codimension 1 in *a unique* space $\overline{K_{\dots}^{\dots}}$ which is up to *a sign* the space $\mathrm{K}_{m_1,\dots,m_{\ell_1}}^{n_1,\dots,n_{\ell_2}} \times \mathrm{K}_{m_1',\dots,m_{\ell_1'}'}^{n_1',\dots,n_{\ell_2'}'}$.

By definition, this sign is equal to the sign in the $\Psi_1 \circ \Psi_2$ before their product in the preCROC $\mathrm{End}(V)$.

Theorem. *The product in $\mho$, defined in this way, is associative.*

Proof. It is clear that the product is associative up to a sign, because the product in the preCROC $\mathrm{End}(V)$ is associative. Only what we need to check are the signs.

We have the canonical projection of CROCs $p\colon \aleph \to \mathrm{Assoc}$. Then, any associative bialgebra is an algebra over the CROC $\aleph$. Consider the tangent space to the space of maps of *pre*CROCs $\mathrm{Der}(\aleph, \mathrm{End}(V))$ at the point, corresponding to the bialgebra above. We define a differential and a product on $\mathrm{Der}(\aleph, \mathrm{End}(V))[-1]$ as follows. Let $\overline{\aleph}$ be the space of the generators of the free CROC $\aleph$, namely, $\overline{\aleph}$ consists from the all cells of codimension 0. Then any element $D \in \mathrm{Der}(\aleph, \mathrm{End}(V)$ is uniquely defined by its restriction to $\overline{\aleph}$. If we have two derivations $D_1, D_2 \in \mathrm{Der}(\aleph, \mathrm{End}(V))$ we can take the composition

$$\overline{\aleph} \to \overline{\aleph}^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V) \tag{20}$$

where the first arrow is the chain differential ∂, the second is $D_1 \otimes D_2$, and the third is the composition in the CROC $\mathrm{End}(V)$. It is clear that this definition of the product in $\mho$ coincides with the definition given above. The advantage of the definition (20) is that here the signs are specified. But we need to prove that this formula gives indeed an associative product.

We do it using the equation $\partial^2 = 0$. Namely, suppose that $D_1 \in \mathrm{End}(V)^{N}_{m_1^1,\dots,m^1_{\ell_1^1};m_1^2,\dots,m^2_{\ell_1^2};\dots;m_1^k,\dots,m^M_{\ell_1^M}}$, $D_2 \in \mathrm{End}(V)^{\ell_2^1,\dots,\ell_2^N}_{\ell_1^1,\dots,\ell_1^M}$, $D_3 \in \mathrm{End}(V)^{n_1^1,\dots,n^1_{\ell_2^1};n_1^2,\dots,n^2_{\ell_2^2};\dots;n_1^N,\dots,n^N_{\ell_2^N}}_{M}$. We want to write down explicitly what follows from the equation

$$\partial^2(\mathrm{K}^{n_1^1+n_2^1+\dots+n^1_{\ell_2^1}+\dots+n_1^N+\dots+n^N_{\ell_2^N}}_{m_1^1+\dots+m^1_{\ell_1^1}+\dots+m_1^M+\dots+m^M_{\ell_1^M}})(D_1\otimes D_2\otimes D_3)=0 \tag{21}$$

One can show that the only interesting boundaries (of codimension 1) in $\partial(\mathrm{K}^{n_1^1+n_2^1+\dots+n^1_{\ell_2^1}+\dots+n_1^N+\dots+n^N_{\ell_2^N}}_{m_1^1+\dots+m^1_{\ell_1^1}+\dots+m_1^M+\dots+m^M_{\ell_1^M}})$ are:

$$\partial_1 = \pm \mathrm{K}^{\ell_2^1+\dots+\ell_2^N}_{m_1^1+\dots+m^1_{\ell_1^1},m_1^2+\dots+m^2_{\ell_1^2},\dots,m_1^M+\dots+m^M_{\ell_1^M}} \times \mathrm{K}_M^{n_1^1,\dots,n^1_{\ell_2^1};\dots;n_1^N+\dots+n^N_{\ell_2^N}} \tag{22}$$

and

$$\partial_2 = \pm \mathrm{K}^{N}_{m_1^1,\dots,m^1_{\ell_1^1};\dots;m_1^M,\dots,m^M_{\ell_1^M}} \times \mathrm{K}^{n_1^1+\dots n^1_{\ell_2^1},\dots,n_1^N+\dots n^N_{\ell_2^N}}_{\ell_1^1,\dots,\ell_1^M} \tag{23}$$

The boundary of the first factor in (22) contains the term $\pm\mathrm{K}^{N}_{m_1^1,\dots,m^1_{\ell_1^1};\dots;m_1^M,\dots,m^M_{\ell_1^M}} \times \mathrm{K}^{\ell_2^1,\dots,\ell_2^N}_{\ell_1^1,\dots,\ell_1^M}$ and the second factor in (23) contains the term $\pm\mathrm{K}^{\ell_2^1,\dots,\ell_1^N}_{\ell_1^1,\dots,\ell_1^M} \times \mathrm{K}_M^{n_1^1,\dots,n^1_{\ell_2^1};\dots;n_1^N,\dots,n^N_{\ell_2^N}}$. These to terms in ∂^2 cancel each other. Thus we get the associativity equation. $\square$

2.5.1 Example

Consider the space $\overline{\mathrm{K}_2^2}$. It is a 1-dimensional space. Its boundary consists from two points, these points are $\mathrm{K}_{1,1}^2 \times \mathrm{K}_2^{1,1}$ and $\mathrm{K}_2^1 \times \mathrm{K}_1^2$. We can write, up to a common sign, $\partial(\overline{\mathrm{K}_2^2}) = \mathrm{K}_2^1 \times \mathrm{K}_1^2 - \mathrm{K}_{1,1}^2 \times \mathrm{K}_2^{1,1}$. This example explains the signs in these Section.

Now we are ready to give the following definitions:

Definition. (i) A strong homotopy bialgebra structure on a vector space V is a map of CROCs $\Upsilon\colon \aleph \to \mathrm{End}(V)$,

(ii) A non-commutative strong homotopy bialgebra structure on a vector space V is a map of *pre*CROCs $\Upsilon\colon \aleph \to \mathrm{End}(V)$.

Remark. Note that in the construction above we also define on the associative algebra $\mho$ a differential, compatible with the product. The differential comes from the "linear" term in the action of ∂ on $\overline{\aleph}$.

3 The Quillen duality and the (non-)commutative deformations

3.1 The Quillen duality

The classical Quillen duality gives two maps $Q_{C\to L}\colon \mathrm{Comm} \to \mathrm{Lie}$ from commutative dg algebras to Lie dg algebras and $Q_{L\to C}\colon \mathrm{Lie} \to \mathrm{Comm}$ from Lie dg algebras to commutative dg algebras which establish the equivalence of the derived categories DComm and DLie. It means, that for a commutative dg algebra $A^\bullet$, the commutative dg algebra $Q_{L\to C} \circ Q_{C\to L}(A)$ is isomorphic to $A^\bullet$ in the derived category, and for a dg Lie algebra $\mathfrak{g}^\bullet$, the dg Lie algebra $Q_{C\to L} \circ Q_{L\to C}(\mathfrak{g}^\bullet)$ is isomorphic in the derived category to $\mathfrak{g}^\bullet$.

These functors $Q_{C\to L}$ and $Q_{L\to C}$ are constructed us follows. For a commutative dg algebra $A^\bullet$, consider the free Lie algebra $\mathrm{Free}((A^\bullet[1])^*)$ generated by the dual space $(A^\bullet[1]]^*$. The product in $A^\bullet$ is a map $S^2(A^\bullet) \to A^\bullet$ where the symmetric square is understood in the graded sense. Then we have the dual map $\delta\colon (A^\bullet)^* \to S^2(A^\bullet)^*$. The map δ can be considered as a map from the generators of the Lie algebra $\mathrm{Free}((A^\bullet[1])^*)$ to the brackets of the generators. It turns out from the associativity of the product in $A^\bullet$ that the map δ can be correctly extended to a differential on $\mathrm{Free}((A^\bullet[1])^*)$ of degree $+1$.

On the other hand, for a dg Lie algebra $\mathfrak{g}^\bullet$, $Q_{L\to C}(\mathfrak{g}^\bullet)$ is by definition the chain complex of the Lie algebra $\mathfrak{g}^\bullet$.

The fact that these to functors define the equivalence of the derived categories DComm and DLie is proven in [Q1,2].

On the other hand, in the same way one can define the functor $Q_{A\to A}$ from associative dg algebras to itself. Namely, for an associative dg algebra $A^\bullet$, consider the free (tensor) associative algebra $T((A^\bullet[1])^*)$ generated by the space $(A^\bullet[1])^*$. The dual map to the

product is a map $\delta\colon (A^\bullet)^* \to \otimes^2(A^\bullet)^*$, and it follows from the associativity of the product in $A^\bullet$ that δ can be continued to a differential in $T((A^\bullet[1])^*)$ by the Leibniz rule. One can prove that $Q^2_{A\to A}(A^\bullet)$ is isomorphic to $A^\bullet$ in the derived category.

Remark. Note that the cohomology of $T((A[1])^*)$ are 0 in all degrees for a degree 0 associative algebra A. Nevertheless, the double application of this construction gives non-trivial cohomology. The point is that the corresponding spectral sequence does not converge to the total cohomology, and we can not use it. Another example of such a situation: Consider the derivations of $T((A^\bullet[1])^*)$. It is clearly the Hochschild cohomological complex of $A^\bullet$, and has non-zero cohomology for many degree 0 algebras.

It is clear that the Quillen duallty for associoative algebras is compatible with the Quillen duality for commutative and for Lie algebras.

3.2 Relation with deformation theory

In deformation theory, the deformations of an object are described via the deformation functor. This is a functor on the Artinian algebras constructed from a dg Lie algebra. This dg Lie algebra is the algebra of the derivations of the object we deform in a higher sense. In each case, we define this dg Lie algebra differently. The general prescription is to replace the object by its resolution and to take the derivations of the resolution. It is a Lie algebra with the differential equal to the bracket with the differential in the resolution (which is a distinguished derivation of the resolution).

On the other hand, the 0-th Lie algebra cohomology of this dg Lie algebra are equal to the commutative algebra of functions on the formal neighborhood of the object in the moduli space of deformations. To make this claim rigorous, we should work carefully with the infinite-dimensional objects, and consider the right completions. For the deformation theory of Riemann surfaces this claim is proved in [F].

Then, it is clear, that the more right object is the extended dg commutative algebra, which is the Quillen dual to the deformation dg Lie algebra. (The cochain complex $Q_{L\to C}(\mathfrak{g}^\bullet)$ computes the Lie algebra cohomology).

In Section 2 of the present paper we constructed a dg associative algebra $\mho$. We can explain its relation with the deformation theory as follows. Suppose that the *extended* (in the sense above) commutative neighborhood in the moduli space is a part of a bigger non-commutative space. The there is a map $p\colon A^\bullet \to A_0^\bullet$ where $A_0^\bullet$ is the extended commutative dg algebra, and $A^\bullet$ is the associative dg algebra. Suppose that $A_0^\bullet = A^\bullet/[A^\bullet, A^\bullet]$. The commutant here should be understood in the sense of the derived functors.

At the moment we do not know how to define the non-commutative algebra $A^\bullet$. We are going to consider this problem in the sequel. But now we think about the algebra $\mho$ as about the Quillen dual to $A^\bullet$, $\mho = Q_{A\to A}(A^\bullet)$. The evidence for this conjecture is that the product of some elements (the "diagonal" elements) in $\mho$ is looks very closely to the Maurer-Cartan equation for bialgebras. This informal conjecture allows us to formulate

another one, more rigorous:

Conjecture

Consider the algebra Quillen dual to $\mho$, $\Omega = Q_{A\to A}(\mho)$. Consider the quotient $A_0^\bullet = \Omega/[\Omega,\Omega]$. Then the 0-th cohomology of $A_0^\bullet$ is isomorphic to the functions on the formal neighborhood of the initial bialgebra in the moduli space of bialgebras. Next, the Quillen dual to the dg commutative algebra $A_0^\bullet$, $\mathfrak{g}^\bullet = Q_{C\to L}(A_0^\bullet)$ is the deformation Lie algebra for deformations of the initial (co)associative bialgebra. It means that the deformation functor associated with this dg Lie algebra, describes the deformations of (co)associative bialgebras.

Remark. The quotient by the commutant in the Conjecture above can be understood in the usual sense because the algebra Ω is free.

3.3 Formality conjectures

In the case when the initial bialgebra is $S(V)$ (a free commutative cocommutative bialgebra), the algebra $\mho$ is quasi-isomorphic to its cohomology as an associative dg algebra. The corresponding Lie algebra $\mathfrak{g}^\bullet$ (constructed in the Conjecture above) in this case is also formal.

In this case, maybe the non-commutative formality (of the algebra $\mho$) is more simple than the formality of the dg Lie algebra $\mathfrak{g}^\bullet$.

Acknowledgements

The author is grateful to Borya Feigin and to Maxim Kontsevich for many discussions. Discussions with Andrey Losev on the CROCs of associative algebras and on Stasheff associahedrons were very useful for me. I am grateful very much to Giovanni Felder for organizing and supervising my postdoc at the ETH (Zurich). I would like to express my gratitude to the ETH and to the IHES (Bures-sur-Yvette) where a part of this work was done for the hospitality and the excellent working conditions.

References

[F] B. Feigin, The conformal field theory from the view of the cohomology theory of Lie algebras, ICM-90 (Kyoto) , Math. Society of Japan, Tokyo, 1990

[K1] M. Kontsevich, Deformation quantization of Poisson manifolds I, preprint q-alg/9709040

[K2] M. Kontsevich, Operads and Motives in Deformation Quantization, preprint math.QA/9904055

[M] M. Markl, A resolution (minimal model) of the PROP for bialgebras, preprint math.AT/0209007

[Q1] D.G. Quillen, Homotopical algebra, Lect. Notes in Math. 43, Springer-Verlag, 1967

[Q2] D.G. Quillen, Rational Homotopy Theory, Ann. of Math., vol.90, 1969

[T] D. Tamarkin, Another proof of M.Kontsevich formality theorem, preprint math.QA/9803025

Dept. of Math., ETH-Zentrum, 8092 Zurich, SWITZERLAND
e-mail: borya@mccme.ru, borya@math.ethz.ch